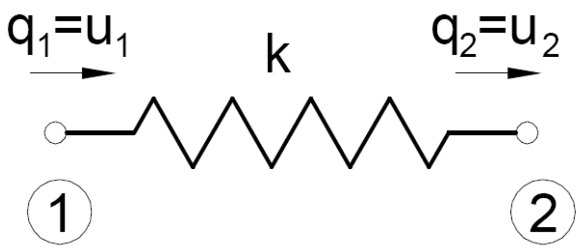
Wydział Mechaniczny Energetyki i Lotnictwa
Zakład Wytrzymałości Materiałów i Konstrukcji

Metoda elementów skończonych (MES1)

Wykład 2C. Element typu sprężyna

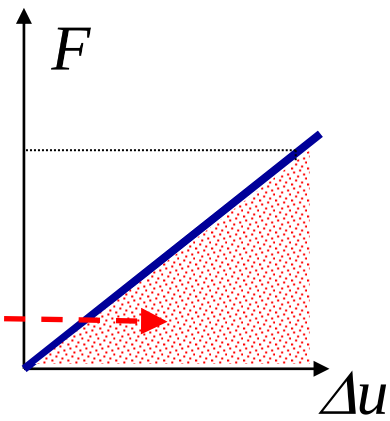
03.2024

Lokalna macierz sztywności sprężyny



wektor parametrów węzłowych:

$$\underset{2\times 1}{\{q\}_e} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e$$



$$F = k \cdot \Delta u = \mathrm{k} \cdot (u_2 - u_1)$$

energia sprężysta elementu:

$$U_e = \frac{1}{2} F \Delta u = \frac{1}{2} k (\Delta u)^2 = \frac{1}{2} k (u_2 - u_1)(u_2 - u_1).$$

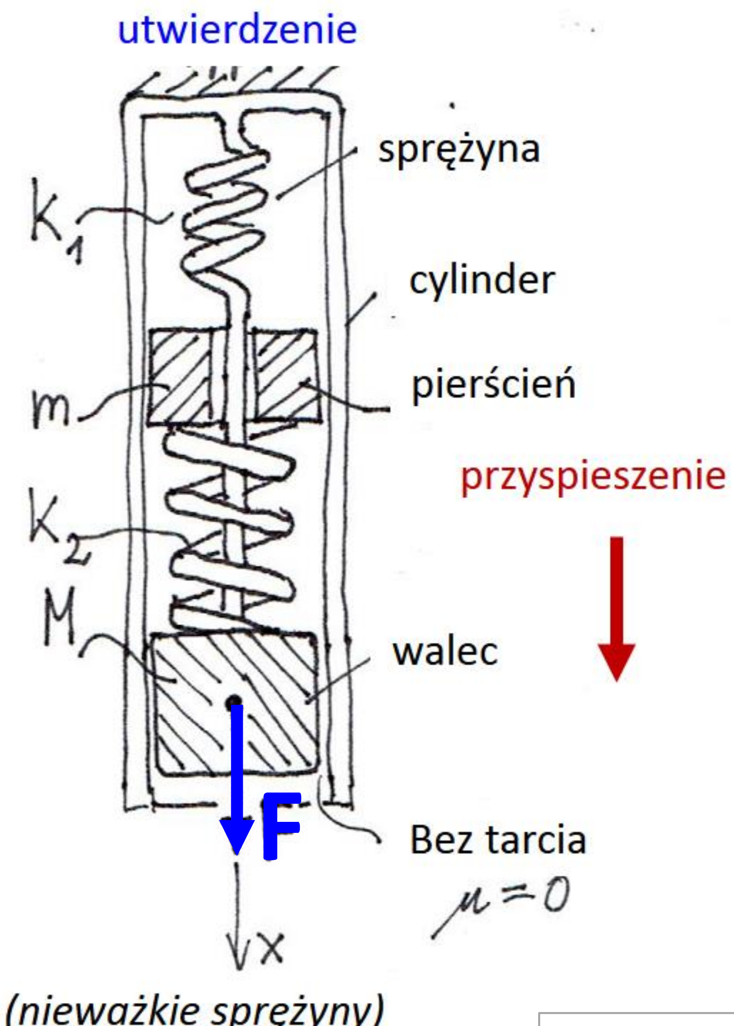
$$U_e = \frac{1}{2} \lfloor u_1, u_2 \rfloor \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\boxed{U_e = \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\}_e}$$

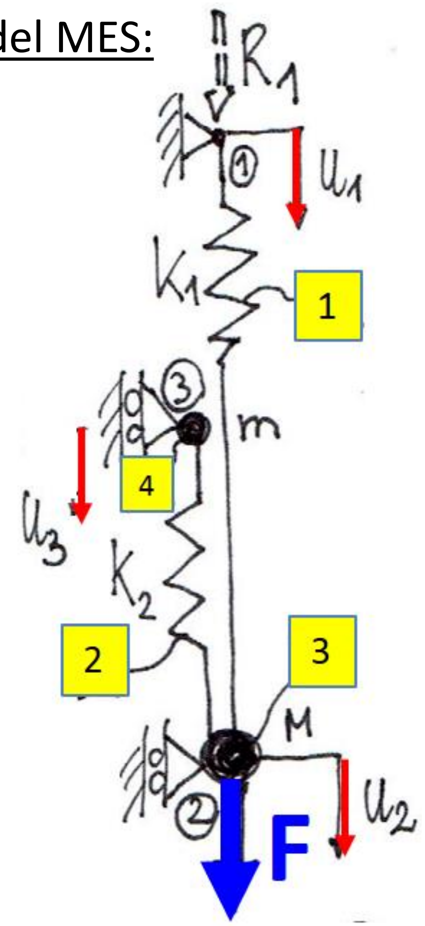
macierz sztywności elementu:

$$\boxed{[k]_e = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}}$$

Przykład: Zbuduj model MES. Znajdź energię sprężystą odkształcenia i energię potencjalną obciążenia. Wyznacz przemieszczenia i reakcje.



Model MES:



■ - Elementy skończone

○ - węzły

$NOE = 4$

$NON = 3$

$n_p = 1$

$NDOF = 3 \cdot 1 = 3$

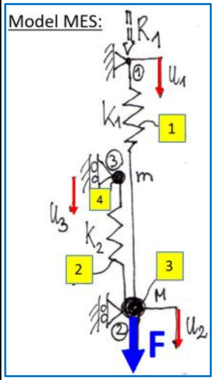
$$\underset{\substack{NDOF \times 1 \\ \shortparallel \\ 3}}{\{q\}} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\underset{3\times1}{\{F\}^n} = \begin{Bmatrix} R_1 \\ F \\ 0 \end{Bmatrix}$$

brak siły w węźle 3

$$U = \frac{1}{2} \underset{1\times3}{\lfloor q \rfloor} \underset{3\times3}{[K]} \underset{3\times1}{\{q\}}$$

$$W = \underset{1\times3}{\lfloor q \rfloor} \cdot \underset{3\times1}{\{F\}}$$



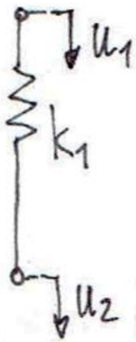
notacja lokalna

Element skończony 1

$$\underset{1\times2}{\lfloor q \rfloor_1} = \lfloor q_1, q_2 \rfloor_1 = \lfloor u_1, u_2 \rfloor_1$$

$$\underset{2\times2}{[k]_1} = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$

1 = 1



2 = 2

(ne) $\underset{1\times3}{\lfloor q \rfloor} = \lfloor \underbrace{u_1, u_2}_{\lfloor q \rfloor_1}, u_3 \rfloor$

$$\lfloor F^x \rfloor_1 = \lfloor F_{11}, F_{21} \rfloor_1 = \lfloor 0, 0 \rfloor_1$$

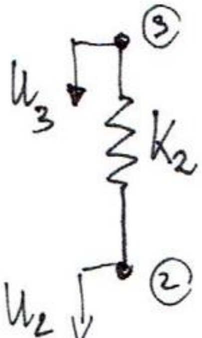
$$[k]_1^* = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{1\times3}{\lfloor F^x \rfloor_1^*} = \lfloor F_{11}, F_{21}, 0 \rfloor = \lfloor 0, 0, 0 \rfloor$$

Element skończony 2

$$\underset{1\times2}{\lfloor q \rfloor_2} = \lfloor q_1, q_2 \rfloor_2 = \lfloor u_3, u_2 \rfloor_2$$

$$\underset{2\times2}{[k]_2} = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

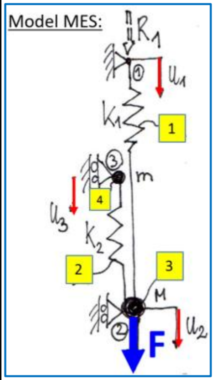


$$\underset{1\times3}{\lfloor q \rfloor} = \lfloor u_1, \boxed{u_2, u_3} \rfloor$$

$$[k]_2^* = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

$$\underset{1\times2}{\lfloor F^x \rfloor_2} = \lfloor F_{12}, F_{22} \rfloor_2 = \lfloor 0, 0 \rfloor_2$$

$$\lfloor F^x \rfloor_2^* = \lfloor 0, F_{22}, F_{12} \rfloor = \lfloor 0, 0, 0 \rfloor$$



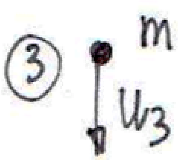
Element skończony 3

$$\underset{1\times 3}{\lfloor q \rfloor} = \lfloor u_1, \boxed{u_2}, u_3 \rfloor \qquad [k]^*_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F^x_3 = M \cdot a_x$$

$$\underset{1\times 3}{\lfloor F^x \rfloor^*_3} = \lfloor 0, F^x_3, 0 \rfloor = \lfloor 0, Ma_x, 0 \rfloor$$

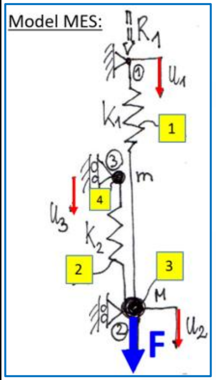
Element skończony 4



$$\underset{1\times 3}{\lfloor q \rfloor} = \lfloor u_1, u_2, \boxed{u_3} \rfloor \qquad [k]^*_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$F^x_4 = m \cdot a_x$$

$$\underset{1\times 3}{\lfloor F^x \rfloor^*_4} = \lfloor 0, 0, F^x_4 \rfloor = \lfloor 0, 0, ma_x \rfloor$$



Globalna macierz sztywności :

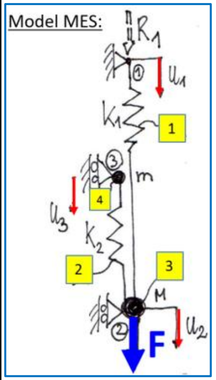
$$\underset{3\times 3}{[K]} = \sum_{e=1}^{4} [k]_e^* = \left[\begin{array}{c|c|c} k_1+0+0+0 & -k_1+0+0+0 & 0+0+0+0 \\ \hline -k_1+0+0+0 & k_1+0+0+k_2 & 0+0+0-k_2 \\ \hline 0+0+0+0 & 0+0+0-k_2 & 0+0+0+k_2 \end{array}\right] =$$

$$= \left[\begin{array}{c|c|c} k_1 & -k_1 & 0 \\ \hline -k_1 & k_1+k_2 & -k_2 \\ \hline 0 & -k_2 & k_2 \end{array}\right]$$

Energia odkształcenia sprężystego:

$$U = \frac{1}{2} \lfloor u_1, u_2, u_3 \rfloor \cdot \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

(Szukamy u_2 i u_3 $(u_1 = 0)$ aby znaleźć wartość U.)



$$\lfloor F \rfloor_e = \lfloor F^X \rfloor_e + \cancel{\lfloor F^p \rfloor_e} = \lfloor F^X \rfloor_e \Rightarrow \lfloor F \rfloor_e^* = \lfloor F^X \rfloor_e^*$$

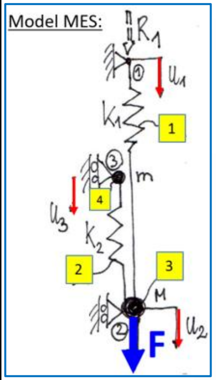
$$W = \underset{1\times 3}{\lfloor q \rfloor} \cdot \underset{3\times 1}{\{F\}}$$

(brak obciążeń powierzchniowych)

$$W = \underset{1\times 3}{\lfloor q \rfloor} \cdot \left(\sum_{e=1}^{4} \{F\}_e^* + \{F\}^n \right) = \underset{1\times 3}{\lfloor q \rfloor} \cdot \left(\begin{Bmatrix} 0+0+0+0 \\ 0+0+Ma_x+0 \\ 0+ma_x+0+0 \end{Bmatrix} + \begin{Bmatrix} R_1 \\ F \\ 0 \end{Bmatrix} \right) =$$

$$= \lfloor u_1, u_2, u_3 \rfloor \cdot \begin{Bmatrix} R_1 \\ Ma_x + F \\ ma_x \end{Bmatrix} \underset{(u_1=0)}{=\!=} u_2 \cdot (Ma_x + F) + u_3 \cdot ma_x$$

(Szukamy u_2 i u_3 aby znaleźć wartość W)



Rozwiązanie:

$$\boxed{V=\frac{1}{2}\lfloor q\rfloor[K]\cdot\{q\}-\lfloor q\rfloor\cdot\{F\}} \longrightarrow \min \longrightarrow \underset{3\times3}{[K]}\cdot\underset{3\times1}{\{q\}}=\underset{3\times1}{\{F\}}$$

$$\boxed{u_1=0} \longrightarrow \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}\cdot\begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix}=\begin{Bmatrix} R_1 \\ Ma_x+F \\ ma_x \end{Bmatrix}$$

$$\underset{2\times2}{[K]}\cdot\underset{2\times1}{\{q\}}=\underset{2\times1}{\{F\}} \longleftarrow$$

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}=\begin{Bmatrix} Ma_x+F \\ ma_x \end{Bmatrix}$$

$$\begin{cases} (k_1+k_2)\cdot u_2 - k_2\cdot u_3 = Ma_x+F \\ -k_2\cdot u_2 + k_2\cdot u_3 = ma_x \end{cases} \Longrightarrow u_2,\ u_3$$

reakcje:

$$\left(\text{1st row of } \underset{3\times3}{[K]}\right)\cdot\underset{3\times1}{\{q\}}=R_1 \Rightarrow R_1=k_1\cdot 0-k_1\cdot u_2+0\cdot u_3=$$
$$=-k_1 u_2$$